Mathematics of Topological Data Analysis (TDA): A Gentle Introduction (Module IV)

Motivation & Syllabus Outline

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February 16, 2026

Contents

1 Module IV: Advanced Topics: Sheaves and Discrete Morse Theory

- Fibers, Persistence, and Sheaves
- Sheaf Cohomology, The Étale Space, and Sections
- Pushforwards and Pullbacks
- Acyclic Partial Matchings
- The Morse Chain Complex and The Equivalence
- Applications to Persistence and Sheaves

1.1 Module Description

This module covers advanced structures for organizing and simplifying topological data. It introduces sheaves, which assign vector spaces (stalks) to simplices and linear maps to face relations, culminating in the construction of sheaf cohomology. It then introduces discrete Morse theory via acyclic partial matchings (combinatorial gradient vector fields). This technique identifies a small set of "critical" simplices, proving that the resulting, much smaller Morse chain complex is chain homotopy equivalent to the original complex.

1.2 Relevance to TDA

While standard persistent homology tracks data over a single real-valued parameter, sheaves provide the optimal data structure for organizing and computing homology across complex, multi-parameter fibers (like mapping data to a torus). Furthermore, TDA on large datasets frequently generates massive simplicial complexes that crash computers; discrete Morse theory serves as a crucial algorithmic optimization, allowing us to drastically shrink the size of the complex while perfectly preserving its homology.

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2 Prerequisites

This module assumes a strong background in topology, particularly metric topology, that is, the theory of metric spaces and their induced topologies. A solid understanding of real analysis is therefore also highly beneficial. QF Academy already offers the following courses that can help build or refresh this background:

- Real Analysis (Module 1): Lecture 2, 3 & 5
- Group Theory, Topology & Manifolds: Topology Module
- Abstract Maths 101 Bootcamp: Topology Module

3 How to Enrol

This module will be rolled out gradually to our most active members starting from the week of February 16, 2026. To enrol, you must be subscribed to any of our existing plans that give you access to the Focused Tracks. Progression through the series is intentional and structured. Enrolment in subsequent modules will be conditional on successfully completing this module. This is a deliberate design choice aimed at discouraging abundance learning where courses are

started but not completed and understanding remains shallow. Members are free to retake this module as many times as needed until they meet the completion requirements. There is no penalty for revisiting the material. The objective is not speed but depth. In 2026, our goal is to significantly increase completion rates and learning outcomes across the academy. This requires focus, sustained effort, and respect for the learning process. By enforcing completion while allowing multiple attempts, we aim to support members in building genuine mathematical fluency rather than simply moving on without a solid foundation.