# Measure Theory and Functional Analysis (Module I)

## Syllabus

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# Course Preview

This is a preview of our upcoming course, **Measure Theory and Functional Analysis (Module I)**, which will be released on **30 April**.

# Course Overview

This module will introduce the foundational concepts of measure theory that are essential for understanding the material in Module II, which will focus on Functional Analysis.

# Topics Covered

1. **Lecture 1: Analysis of the Riemann Integral** – Intuition behind the Riemann integral; infimum and supremum; lower and upper rectangular bounds; lower and upper integrals; precise definition of Riemann integral; example: example of calculating the Riemann integral of $x^2$; fundamental theorem of calculus; proof that Thomae's function is integrable; example of a non-integrable function.

2. **Lecture 2: Introduction to $\sigma$-Algebras** – Definition of an algebra (of sets in some superset); definition of a $\sigma$-algebra; $\sigma$-algebras as a disjoint union of sets; intersection of finitely many $\sigma$-algebras is a $\sigma$-algebra; generating $\sigma$-algebras from sets; Borel $\sigma$-algebras; product $\sigma$-algebras and their generating sets; Borel $\sigma$-algebra on $\mathbb{R}^n$ as the product of Borel $\sigma$-algebras on $\mathbb{R}$; intuition behind measures; Vitali sets as examples of non-measurable sets.

3. **Lecture 3: Measures** – Formal definition of a measure, measure space; examples of measures; proof that measures are monotone; proof that measures are subadditive; proof that measures have continuity from below/above; null (measure zero) sets; truth 'almost everywhere' (a.e.); complete measures; completions of measures.

4. **Lecture 4: Outer Measures** – Definition of outer measures; constructing outer measures with covering sets; $\mu^*$-measurable sets; Carathéodory's Theorem and its proof; defnintion of pre-measures; inducing outer measures with pre-measures.

5. **Lecture 5: Borel Measures on $\mathbb{R}$** – Defining a pre-measure from a right-continuous increasing function; increasing right continuous functions uniquely define Borel measures and vice-versa; measures of sets as infima of measures of open covers; measures of sets as suprema of measures of compact subsets; definition of the Lebesgue measure; Lebesgue measures under dilations and translations; example of Lebesgue measure of rationals in an interval.

6. **Lecture 6: Measurable Functions** – Recap on preimages of functions; preimages of $\sigma$ algebras are $\sigma$-algebras; definition of measurable functions; compositions of measurable functions; measurability by pre-images of generating sets; definition of Lebesgue measurable functions; Lebesgue measurable sets as completion of the Borel $\sigma$-algebra;

equivalent conditions to measurability for real-valued functions; measurability under summation and multiplication; definition of the extended real line and its Borel $\sigma$-algebra; positive and negative parts of functions; characteristic (indicator) functions and simple functions; measurable functions as pointwise limits of simple functions.

7. **Lecture 7: Integration of Non-Negative Functions** – Definition of integral of a positive simple function with respect to a measure (over entire space or some other measurable set); basic properties of integration with respect to a measure; integrals with respect to measures defining new measures; definition of integrals of non-simple measurable functions; the Monotone Convergence Theorem and its proof; integrals of sums of (up to countably infinitely many) non-negative measurable functions; Fatou's Lemma; Dominated Convergence Theorem.

8. **Lecture 8: Lebesgue vs Riemann Integration** – Recap on Riemann integration; proof of equivalence of Riemann integrability and continuity almost everywhere; example with indicator function of rationals; example with Thomae's function.

9. **Lecture 9: Modes of Convergence** – Convergence almost everywhere of a sequence of functions on a measure space; convergence in measure of a sequence of functions on a measure space; convergence in $L^p$ of a sequence of functions on a measure space; convergence in $L^1$ implies convergence in measure; convergence in measure implies subsequence converges a.e.; examples on simple functions; Egoroff's Theorem.

10. **Lecture 10: The Fubini-Tonelli Theorem** – Product measures recap; measurable rectangles; finite disjoint unions of rectangles generate a $\sigma$-algebra on the product space; the product measure; cross sections of product spaces; $\sigma$-finite measure spaces; product measure is equal to integrating one cross section over the other coordinate; the Fubini-Tonelli Theorem and its proof; example of using Fubini-Tonelli on a function.