Measure Theory and Functional Analysis (Module I)

Syllabus

`quantumformalism.academy`

Course Overview

This module will introduce the foundational concepts of measure theory that are essential for understanding the material in Module II, which will focus on Functional Analysis.

Topics Covered

1. **Lecture 1: Analysis of the Riemann Integral** – Intuition behind the Riemann integral; infimum and supremum; lower and upper rectangular bounds; lower and upper integrals; precise definition of Riemann integral; example: example of calculating the Riemann integral of x^2; fundamental theorem of calculus; proof that Thomae's function is integrable; example of a non-integrable function.

2. **Lecture 2: Introduction to σ-Algebras** – Definition of an algebra (of sets in some superset); definition of a σ-algebra; σ-algebras as a disjoint union of sets; intersection of finitely many σ-algebras is a σ-algebra; generating σ-algebras from sets; Borel σ-algebras; product σ-algebras and their generating sets; Borel σ-algebra on $\mathbb{R}^n$ as the product of Borel σ-algebras on $\mathbb{R}$; intuition behind measures; Vitali sets as examples of non-measurable sets.

3. **Lecture 3: Measures** – Formal definition of a measure, measure space; examples of measures; proof that measures are monotone; proof that measures are subadditive; proof

that measures have continuity from below/above; null (measure zero) sets; truth 'almost everywhere' (a.e.); complete measures; completions of measures.

4. **Lecture 4: Outer Measures** – Definition of outer measures; constructing outer measures with covering sets; μ^*-measurable sets; Carathéodory's Theorem and its proof; definition of pre-measures; inducing outer measures with pre-measures.

5. **Lecture 5: Borel Measures on $\mathbb{R}$** – Defining a pre-measure from a right-continuous increasing function; increasing right continuous functions uniquely define Borel measures and vice-versa; measures of sets as infima of measures of open covers; measures of sets as suprema of measures of compact subsets; definition of the Lebesgue measure; Lebesgue measures under dilations and translations; example of Lebesgue measure of rationals in an interval.

6. **Lecture 6: Measurable Functions** – Recap on preimages of functions; preimages of σ-algebras are σ-algebras; definition of measurable functions; compositions of measurable functions; measurability by pre-images of generating sets; definition of Lebesgue measurable functions; Lebesgue measurable sets as completion of the Borel σ-algebra; equivalent conditions to measurability for real-valued functions; measurability under summation and multiplication; definition of the extended real line and its Borel σ-algebra; positive and negative parts of functions; characteristic (indicator) functions and simple functions; measurable functions as pointwise limits of simple functions.

7. **Lecture 7: Integration of Non-Negative Functions** – Definition of integral of a positive simple function with respect to a measure (over entire space or some other measurable set); basic properties of integration with respect to a measure; integrals with respect to measures defining new measures; definition of integrals of non-simple measurable functions; the Monotone Convergence Theorem and its proof; integrals of sums of (up to countably infinitely many) non-negative measurable functions; Fatou's Lemma; Dominated Convergence Theorem.

8. **Lecture 8: Lebesgue vs Riemann Integration** – Recap on Riemann integration; proof of equivalence of Riemann integrability and continuity almost everywhere; example with indicator function of rationals; example with Thomae's function.

9. **Lecture 9: Modes of Convergence** – Convergence almost everywhere of a sequence of functions on a measure space; convergence in measure of a sequence of functions on a measure space; convergence in L^1 of a sequence of functions on a measure space; convergence in L^1 implies convergence in measure; convergence in measure implies subsequence converges a.e.; examples on simple functions; Egoroff's Theorem.

10. **Lecture 10: The Fubini-Tonelli Theorem** – Product measures recap; measurable rectangles; finite disjoint unions of rectangles generate a σ-algebra on the product space; the product measure; cross sections of product spaces; σ-finite measure spaces; product measure is equal to integrating one cross section over the other coordinate; the Fubini-Tonelli Theorem and its proof; example of using Fubini-Tonelli on a function.