QFA MATHEMATICAL FOUNDATIONS OF QUANTUM COMPUTING SYLLABUS FOR MODULES 1 AND 2

Module 1

(1) **Vector Spaces and Subspaces:** Definition of vector spaces; Examples - $\mathbb{C}^n$ and Cartesian products; Subspaces; Trivial and proper subspaces; Linear spans; Vector spaces of matrices.

(2) **Inner Products:** Defninition of inner products; Proof of conjugate linearity in second variable; The standard inner product for $\mathbb{C}^n$.

(3) **Norms, and Norms from Inner Products:** Definition of norms and normed spaces; The operator norm for $M_n(\mathbb{C})$; The supremum norm for $\ell_\infty(\mathbb{C})$; The norm for inner product spaces; The Euclidean norm; The Cauchy-Schwarz inequality; The parallelogram law; The condition for a norm to define an inner product.

(4) **Banach and Hilbert Spaces:** Sequences; Convergent sequences; Cauchy sequences; Convergent implies Cauchy proof; Example of a non-convergent Cauchy sequence; Closed and open subspaces (for normed spaces); Completeness; Definition of Banach, Hilbert spaces; Proof that $\mathbb{C}^n$ is a Hilbert space.

(5) **Orthogonality, Orthonormality, and Dimension:** Definition of orthogonal and orthonormal pairs and systems; Proof that orthogonal vectors are linearly independent; Definition of basis (orthogonal basis, orthonormal basis); Schauder and Hamel bases; Every Hilbert space has an orthonormal basis; Examples - bases of $\mathbb{C}^3$ and $\ell_2(\mathbb{N})$; Definition of cardinality; Definition of dimension of Hilbert spaces; Denseness and separability; Parsavel's identity; Bijectivity and isometric isomorphisms; All n-dimensional Hilbert spaces are isometrically isomorphic

(6) **The Gram-Schmidt Process and Change-of-Basis Matrices:** Gram-Schmidt orthonormalisation process, and proof that it works; Example of Gram-Schmidt on $\mathbb{R}^3$; Definition of change-of-basis matrix; How to construct a change-of-basis matrix; Motivation and example of change-of-basis matrices for $\mathbb{R}^2$.

(7) **Dual Spaces and the Riesz Representation Theorem:** Definition of functionals, continuous functionals, (algebraic, Banach) dual spaces; Continuity vs boundedness of functionals; Completeness of duals of normed spaces; The Riesz Representation Theorem; Hilbert spaces are (up to isomertric isomorphism) the same as their duals; Dirac notation; Example - a Banach space not the same as its dual.

(8) **Tensor Products:** Bilinear maps; Definition of tensor product of vectors and vector spaces; Rank-one tensors; Properties of tensor products (bilinearity, distributivity); Subspaces of tensor product spaces; Bases of tensor product spaces; Dimension of tensor product spaces; The universal property; The inner product of a tensor product of Hilbert spaces; Separable and entangled tensors; Example - an entangled tensor; Tensor products of operators, definition and as matrices.

Module 2

(1) **Operators Between Banach Spaces:** Definition of operators; Equivalence of continuity and boundedness; The operator norm; Sets of bounded operators is a normed vector space.

(2) **Important Types of Operator:** Riesz Representation Theorem recap; Adjoint operators; Identity and zero operators; Hermitian operators (and their eigenvectors/values); Isometries; Isomorphisms; Unitary operators; Normal operators; Rank-one, finite-rank operators; Compact operators; Orthogonal projections (idempotence and self-adjointness).

(3) **Spectral Theory in Finite Dimensions:** Kernels of operators and injectivity; The Rank-Nullity Theorem; Spectrums of operators; Existence of eigenvalues in finite dimensions (and counterexample in infinite dimensions);Eigenvalues of unitary operators; The Spectral Theorem for Hermitian operators; Decomposition of the identity; Decomposition of a Hermitian operator.

(4) **One-Qubit Systems:** The state space of a qubit; Operations on a qubit; Common gates; Measurements(of Z); Measurements of arbitrary observables; State collapse via decomposition of observable; Expectations of measurements; Example - measurement of two different observables; Global phases and indistinguishability; The Bloch sphere; Rotation gates.

(5) **Multiple-Qubit Systems:** The state space for many qubits; Entangled states; Example - proving that a state is entangled;

Operations on multiple qubits; Common gates for multiple qubits; Measurements of multiple-qubit states with respect to observables; Circuit diagrams; Example -creating a bell state.

(6) **Mixed States and Density Matrices:** Definition of pure and mixed states; Outer products; Density matrices; Proof that density matrices are Hermitian, trace 1, positive semidefinite; Applying gates to mixed states; Measurements of mixed states with respect to observables; Example - difference between mixing states and superposition.

(7) **Noise and State/Gate Preparation:** Definition of noise; Definition of Quantum ChannelS; Kraus' Theorem and Kraus Operators; Fidelity; Example - effects of noise on constructing a Bell state; State/Gate preparation via Universal gatesets; The Solovay-Kitaev Theorem; Proving that a given gateset is universal for one-qubit computations.

(8) **Classical-to-Quantum Data Embeddings and the Hadamard Test:** Motivation for quantum embeddings; Angle embedding definition; Fidelity of angle embedded states; Why Z can not be the axis for angle embedding; Pros and cons of angle embedding; Basis embedding definition; Fidelity of basis embedded states - individual numbers and sets of numbers; Amplitude embedding definition; Pros and cons of amplitude embedding; The Hadamard test example.