QFA REAL ANALYSIS
MODULE 2 SYLLABUS

1. Real-Valued Functions: IVT, EVT, and Modes of Convergence

- State and prove the Intermediate Value Theorem (IVT), giving an example and a diagram.
- Use the IVT to prove the Extreme Value Theorem (EVT).
- Apply the EVT to give an alternative proof that uniformly continuous functions on bounded sets have bounded image.
- Define pointwise and uniform convergence for sequences of real-valued functions, and show that uniform convergence implies pointwise convergent, however the converse does not hold.
- Prove that uniform limits of sequences of continuous functions are continuous.
- State, prove, and demonstrate Dini's Theorem, combining pointwise monotonicity and pointwise convergence to obtain uniform convergence.
- Give the Weierstrass M-Test for convergence of series of functions.
- Briefly introduce L_p convergence of sequences of functions.

2. The Derivative

- Define differentiability, give examples, and show that although differentiablility implies continuity, non-differentiable functions may still be continuous.
- State and prove the rules for differentiation of sums, scalar multiples, and products.
- Prove the Chain Rule for differentiating compositions of functions.
- State and prove Rolle's Theorem, with examples.
- Derive the Mean Value Theorem (MVT) from Rolle's Theorem and verify with examples.
- State L'Hôpital's Rule, proving the case for the 0/0 indeterminate forms. Give a simple example.
- State Taylor's Theorem, define Taylor polynomials, and examine the accuracy of such approximations for a special case.

- State and prove the Inverse Function Theorem for local invertability and differentiability of the inverse, with a sketch proof.
- Show that uniform convergence of derivatives implies differentiability of the limit function, with an example.
- Introduce the Newton-Raphson Method for root-finding, giving a demonstration.

3. The Integral

- Define partitions of an interval, in order to define lower and upper Darboux sums $L(f, P)$ and $U(f, P)$.
- Define Riemann integrability via Darboux sums as equality of $\sup L(f, P)$ and $\inf U(f, P)$.
- Prove that continuous functions on compact intervals are integrable. Discuss counterexamples of integrable but discontinuous functions.
- State and prove basic properties of the integral, namely linearity, additivity over subintervals, inequality preservation, and the triangle inequality.
- State and prove the Mean Value Theorem for Integrals.
- State and prove the Fundamental Theorem of Calculus (FTC), both Part I and Part II. Proof sketches use the Mean Value Theorem for Integrals.
- Introduce improper integrals as integrals with either unbounded domains or asymptotic integrands. Provide some convergence criteria.
- Demonstrate a few simple examples of improper integrals.
- Prove that uniform convergence of integrable functions allows interchange of limit and integral.
- Give counterexamples to show that pointwise convergence is insufficient to interchange limits and integrals.

4. Abstraction to Metric Spaces

- Define metric spaces and provide key examples (Euclidean metric, discrete metric, supremum norm metric).
- Work through an example showing that the supremum metric d_∞ on $C([a, b])$ is indeed a metric.
- Define norms, and show that every norm induces a metric. Give examples of metrics derived from norms and show that the discrete metric is not norm-induced.
- Redefine several key topological definitions that we already knew for $\mathbb{R}$, in the more abstract setting of metric spaces.

- Show that the Heine–Borel theorem fails in general metric spaces, with counterexample using the discrete metric.
- Define isometries and contractions. Provide examples in Euclidean space and $C([0,1])$.
- State and give a sketch proof that every metric space has a canonical completion, defined as the closure of a certain isometric image of the original space.
- Prove the Contraction Mapping Theorem (aka. Banach Fixed Point Theorem).
- Define Lipschitz maps and show that Lipschitz continuity implies uniform continuity.
- Apply the Contraction Mapping Theorem to prove the Picard–Lindelöf Theorem, which establishes local existence and uniqueness of solutions to initial value problems in ODE theory.

5. The Stone-Weierstrass Theorem

- State the Weierstrass Approximation Theorem: polynomials are dense in $C([a,b])$. Detail that Bernstein polynomials can always be used as arbitrarily close approximations of continuous functions, giving a constructivist method to prove the Theorem.
- Define algebras of functions, subalgebras, and the concept of separating points. Show that $C(K)$ is an algebra when $K \subseteq \mathbb{R}$ is compact, and that the space of polynomials separate points.
- Review topological preliminaries needed for Stone–Weierstrass, including the open cover definition of compactness and the preimages definition of continuity.
- State and prove the Stone–Weierstrass Theorem for compact $K \subseteq \mathbb{R}$: if a subalgebra of $C(K)$ contains constants and separates points, then its closure is $C(K)$.
- Apply Stone–Weierstrass to show that polynomials are dense in $C([a,b])$, recovering the Weierstrass theorem as a special case.
- Define trigonometric polynomials and show that they form a subalgebra of $C_{2\pi}(\mathbb{R})$ containing constants.
- Handle periodic domains by identifying $C_{per}[0,2\pi]$ with $C(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the circle (a compact Hausdorff space).
- Prove that trigonometric polynomials separate points in $[0,2\pi)$. Apply Stone–Weierstrass to deduce density of trigonometric polynomials in $C_{per}[0,2\pi]$.
- Introduce Fourier series as a constructive method to approximate periodic functions by trigonometric polynomials. Define Fourier coefficients, and write down the Fourier expansion.