QFA REAL ANALYSIS
MODULE 1 SYLLABUS

1. What are the Real Numbers?

- Introduce axiomatic set theory, specifically via the ZFC axioms.
- Examine the intuition behind the natural numbers and their properties.
- Define partial and well orders on sets.
- Define (von Neumann) ordinals, and via this definition, the natural numbers.
- Derive the definition of integers from that of natural numbers, and from there derive a definition for rationals.
- Construct real numbers via Dedekind cuts of rationals.
- Explore cardinals as generalisations of natural numbers, and contrast the concept to ordinals in the infinite case.
- Discuss countable vs. uncountable sets.
- Prove the uncountability of $\mathbb{R}$ using Cantor's diagonal argument.
- Discuss Cantor's Power Set Theorem, hierarchy of uncountable cardinals, and the continuum hypothesis.

2. Sequences and Series of Reals

- Define convergence, divergence, monotonicity, Cauchyness, and boundedness for sequences in $\mathbb{R}$.
- Prove the uniqueness of limits for convergent sequences in $\mathbb{R}$.
- State the Monotone Convergence Theorem.
- Contrast Cauchy and convergent sequences, explaining that completeness of $\mathbb{R}$ gives equivalence of the concepts.
- Prove the Bolzano-Weierstrass Theorem (bounded sequences in $\mathbb{R}$ have convergent subsequences).
- Define $\limsup$ and $\liminf$, and relate them to convergence.
- Define infinite series as limits of partial sums.
- Introduce absolute and conditional convergence.
- Present the Cauchy convergence criterion for series.
- Present and apply key convergence tests: comparison test, ratio test, and alternating series test.

3. Topology of the Real Line

- Define open and closed subsets of $\mathbb{R}$ using open balls and limit points.
- Explore the closure of sets and the notion of dense subsets.
- Reframe definitions of sequence convergence and boundedness using open balls.
- Prove equivalence: a set is open if and only if its complement is closed.
- Explore the behaviour of unions and intersections for open and closed sets.
- Formally define topologies to abstract the discussion.
- Introduce (sequential) compactness and prove the Heine-Borel theorem.
- Define connectedness and characterise connected/disconnected sets in $\mathbb{R}$ as intervals.
- Give examples illustrating perfect and totally disconnected sets.

4. The Cantor Middle-Thirds Set

- Introduce fractals and self-similarity; present examples.
- Define and construct the Cantor middle-thirds set $\mathcal{C}$ as infinite intersection of sets of intervals.
- Prove equivalence of three different definitions of $\mathcal{C}$.
- Prove the Cantor set is compact, perfect, nowhere dense, totally disconnected, and uncountable.
- Briefly introduce related notions, such as scattered and meagre sets.
- Define and compute fractal dimensions, specifically similarity and box-counting dimension for the Cantor set.

5. Functions: Limits and Continuity

- Give the ϵ-δ and sequential definitions of limits for real functions.
- Apply limit laws for finite and infinite limits (sum, product, quotient, and indeterminate forms).
- Present three equivalent definitions of continuity at a point and on a domain.
- Understand left and right continuity, and their relation to overall continuity at a point.
- Define and distinguish between continuity and uniform continuity, with examples.
- Examine three types of discontinuity.

- See that products, quotients, compositions, sums, and scalar multiples of continuous functions are continuous.
- Prove that a continuous function on a compact set is uniformly continuous
- Prove that the image of a continuous function on a compact set is compact.
- Discuss path-connectedness and its equivalence with connectedness in $\mathbb{R}$.