QFA REAL ANALYSIS
DRAFT SYLLABUS

1. What are the Real Numbers?

- Intuition behind axiomatic set theory; the ZFC axioms; rules of inference.
- Intuition behind natural numbers, with respect to their ordinal and cardinal behaviour.
- Well-ordered sets: the well-ordering theorem; the axiom of choice; Zorn's lemma.
- Ordinal sets defined via well-orders.
- von Neumann's definition of the ordinals. Constructing the natural numbers. Operations on the natural numbers.
- Defining the integers from the naturals. Operations on the integers.
- Defining the rationals from the integers.
- Defining the real numbers as Dedekind cuts of the rationals.
- The Cantor-Bernstein-Schroder theorem. Ordering sets via injections. Cantor's power set theorem.
- Cardinals as equivalence classes of sets under bijectivity. Comparison to ordinals.

2. Topology of the Real Line

- Open and closed sets - Definition with neighbourhoods; complement relationship; union and intersection properties.
- Interior/exterior points; isolated points.
- Limit points and accumulation points; closure as union of set with its limit points; characterisation using sequences.
- Compact sets - Heine-Borel theorem: a set is compact if and only if it is closed and bounded; sequential compactness.
- Connected sets - definition via separation into disjoint open sets; intervals as connected sets.
- Cantor set - construction, properties (closed, bounded, uncountable, zero measure, perfect).

3. Sequences and Series

- Convergence and divergence of sequences - ϵ/δ-definition of limits; uniqueness of limits.
- Monotone convergence theorem - bounded monotonic sequences converge; applications to series and continued fractions.
- Compactness.
- Bolzano-Weierstrass theorem - every bounded sequence has a convergent subsequence.
- Cauchy sequences and completeness in $\mathbb{R}$.
- $\limsup$ and $\liminf$ - relationship to convergence; cluster points.
- Series convergence tests - comparison; ratio; alternating series.
- Absolute and conditional convergence - Relationship between absolute and conditional convergence; examples.

4. Limits and Continuity

- Limits of functions - ϵ-δ definition; relationship to sequences; limit laws.
- Left and right limits of functions - (two-sided) limits. Discontinuities.
- Uniform continuity - definition and distinction from pointwise continuity; continuous functions on compact sets are uniformly continuous.
- Intermediate Value Theorem - continuous functions on intervals take all intermediate values; applications to root-finding.
- Extreme Value Theorem - continuous functions on compact sets attain maximum and minimum values.

5. Differentiation

- Definition of derivative - limit of difference in y vs difference in x; left and right derivatives with left and right limits.
- Differentiability implies continuity - proof and geometric understanding; converse is false.
- Mean value Theorem and Rolle's theorem - statements, proofs, and applications.
- L'Hopital's rule for derivatives of rational functions.
- Taylor's theorem with remainder - Taylor polynomials, Lagrange and integral forms of remainder.
- Inverse function theorem - conditions for existence and differentiability of inverse functions.

6. Integration

- Riemann integration - definition via Riemann sums; geometric interpretation.
- Darboux sums and integrability conditions - upper and lower sums; Riemann integrability criterion; integrability of continuous and monotonic functions.
- Fundamental Theorem of Calculus - relationship between differentiation and integration.
- Improper integrals -convergence and divergence; comparison tests; principal values.
- Uniform convergence and integration - when limits and integrals can be interchanged.

7. Sequences and Series of Functions

- Pointwise vs. uniform convergence - definitions and examples showing the distinction; proof that uniform convergence is stronger.
- Weierstrass M-test - sufficient condition for uniform convergence of series of functions.
- Continuity of uniform limits - uniform limit of continuous functions is continuous; counterexamples for pointwise convergence.
- Term-by-term differentiation and integration - conditions to perform operations termwise.
- Power series - radius of convergence (ratio and root tests); uniform convergence on compact subsets of interval of convergence.
- Stone-Weierstrass theorem for polynomials.

8. Metric Spaces

- Definition and examples - Distance functions, triangle inequality; $\mathbb{R}^n$ with various metrics, function spaces.
- Convergence in metric spaces - Generalization of sequence convergence; relationship to topology.
- Completeness and completion - Cauchy sequences in metric spaces; completion process.
- Contraction mapping theorem - Fixed point theorem; applications to differential equations and iterative methods.