A CRASH COURSE ON GROUP THEORY, TOPOLOGY & MANIFOLDS

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Topology and Manifolds

Session 1: Introduction to Topology.

- Fundamental concepts like open and closed sets, continuous functions, compactness, and connectedness.
- Lots of examples, with a focus on how these ideas relate to Lie groups.
- **Quiz:** Simple topology scenarios (e.g., Is this set open/connected?)

Session 2: Foundations of Smooth Manifolds.

- Introduce the structure of smooth manifolds, charts, and atlases. Embedded submanifolds and the implicit function theorem.
- Why *smoothness* matters for Lie groups, rather than just working with topological groups.
- **Quiz:** Identifying manifolds and non-manifolds from examples.

Session 3: Tangent Spaces and Smooth Maps.

- Focuses on the differential structure of manifolds, tangent spaces, and smooth maps. Derivations and tangent vectors at a point, the tangent space.
- Connections to Lie groups, like the exponential map (briefly covered now, more in-depth later on in the course).
- **Quiz:** Calculating tangent vectors or Jacobian matrices.

Session 4: Applications to Lie Groups.

- Bridge the material from topology and smooth manifolds directly to Lie theory.
- Examples of topological properties of matrix Lie groups (e.g., $SL(n, R), O(n)$)
- **Quiz:** diagnostic quiz covering all sessions

Group Theory

Session 1: Fundamentals of Group Theory.

- Definition of a group and motivating examples (symmetries, number systems)
- Group axioms with geometric and algebraic interpretations
- Subgroups, cosets, and quotient groups
- Homomorphisms and Isomorphisms
- **Quiz:** identifying groups from given operations, find subgroups of matrix groups, determine if maps between groups are homomorphisms

Session 2: Group Actions and Representations.

- Group actions: definition, orbits, stabilizers
- Visualizing actions of D_{2n}, $O(n)$, and $SO(n)$ on Euclidean space.
- Cayley's theorem and examples of permutation representations
- Linear representations of groups
- **Quiz:** Compute orbits and stabilizers, determine if representations are faithful, decompose group actions into simpler parts

Session 3: Structure Theory and Matrix Groups.

- Normal subgroups and simple groups
- Jordan-Hölder theorem and group decompositions
- Special classes of groups (Abelian, nilpotent, solvable)
- **Quiz:** diagnostic quiz covering all sessions