Stochastic Processes (Random Walks): A Rigorous Introduction (Module II)

Motivation & Syllabus Outline

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Contents

1 Module II: Random Variables and Expectation

1.1 Scope

With the measure-theoretic foundations in place, this module introduces measurable functions (random variables) and constructs the mathematical expectation as a Lebesgue integral. We also cover fundamental inequalities and the core convergence theorems that allow us to interchange limits and expectations.

1.2 Learning outcomes

By the end of Module II, participants should be able to:

- Rigorously define random variables as measurable mappings and derive their induced probability distributions (pushforward measures).
- Construct the Lebesgue integral in stages (simple functions to non-negative to general) and understand its advantage over the Riemann integral.
- State and apply fundamental inequalities (Markov, Chebyshev, Cauchy-Schwarz, and Jensen).
- Apply the Monotone Convergence Theorem (MCT), Dominated Convergence Theorem (DCT), and Fatou's Lemma to evaluate limits of expectations.

1.3 Lectures

1. **Lecture IV: Random Variables and Distributions**

   Topics: Measurable functions, generated σ-algebras $\sigma(X)$, pushforward measures, cumulative distribution functions (CDFs), and independence of random variables.

2. **Lecture V: The Lebesgue Integral and Expectation**

   Topics: Integration of simple functions, extension to non-negative measurable functions, integration of general measurable functions, properties of expectation.

3. **Lecture VI: Inequalities and Limit Theorems**

   Topics: Markov's and Chebyshev's inequalities, Jensen's inequality, modes of convergence (almost surely, in probability, in L^p), MCT, DCT, and Fatou's Lemma.

1.4 Suggested checkpoints

- Be able to rigorously show that a given transformation of a random variable is still measurable.
- Be able to quickly identify whether MCT, DCT, or neither should be used to evaluate $\lim_{n\to\infty} \mathbb{E}[X_n]$.