



# Stochastic Processes (Random Walks): A Rigorous Introduction (Module III)

Motivation & Syllabus Outline

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# 1 Module III: Conditional Expectation and Martingales

## 1.1 Scope

This module moves beyond elementary conditional probability to rigorously define conditional expectation with respect to a  $\sigma$ -algebra. It then introduces filtrations and discrete-time martingales—the foundational models for fair games and noise-driven systems—culminating in the powerful Optional Stopping and Convergence theorems.

## 1.2 Learning outcomes

By the end of Module III, participants should be able to:

- Define conditional expectation rigorously using  $\sigma$ -algebras and the Radon-Nikodym theorem (or  $L^2$  projections).
- Fluently apply the algebraic properties of conditional expectation (Tower Property, “taking out what is known”).
- Model evolving information using filtrations and adapted stochastic processes.
- Identify martingales, submartingales, and supermartingales, and analyze them using stopping times.
- Apply Doob’s Optional Stopping Theorem and the Martingale Convergence Theorem to solve classical problems like the Gambler’s Ruin.

## 1.3 Lectures

### 1. Lecture VII: Rigorous Conditional Expectation

*Topics:* Information and  $\sigma$ -algebras, conditional expectation as an  $L^2$  projection, formal definition via partial averaging, key properties (linearity, monotonicity, tower property, independence).

### 2. Lecture VIII: Filtrations and Martingales

*Topics:* Filtrations as information flow, adapted and predictable processes, definition of discrete-time martingales, random walks, Doob decomposition.

### 3. Lecture IX: Stopping Times and Convergence

*Topics:* Definition of stopping times, stopped processes, Doob’s Optional Stopping Theorem, upcrossing inequality, Martingale Convergence Theorem.

## 1.4 Suggested checkpoints

- Be able to intuitively explain  $\mathbb{E}[X|\mathcal{G}]$  as the “best guess” of  $X$  given the information in  $\mathcal{G}$ , and prove it minimizes mean squared error.
- Given a stochastic process, be able to construct a filtration  $\mathcal{F}_n$  and rigorously verify the martingale condition  $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ .