Stochastic Processes (Random Walks): A Rigorous Introduction (Module IV)

Motivation & Syllabus Outline

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Contents

1 Module IV: Markov Chains and Random Walks

1.1 Scope

This final module introduces the Markov property—a mathematical formalization of "memorylessness." We explore discrete-time Markov chains on countable state spaces, rigorously classify their long-term behavior, and apply these tools to deeply analyze simple and general random walks on $\mathbb{Z}^d$.

1.2 Learning outcomes

By the end of Module IV, participants should be able to:

- Formulate stochastic systems using the strong and weak Markov properties and transition matrices.
- Classify states within a Markov chain (transient, null recurrent, positive recurrent, periodic, ergodic).
- Compute invariant (stationary) measures and prove the Ergodic Theorem for Markov chains.
- Analyze random walks on lattices, including computing return probabilities and solving the Dirichlet problem for harmonic functions.

1.3 Lectures

1. **Lecture X: The Markov Property and Transition Matrices**

   Topics: Definition of the Markov property, transition probabilities, Chapman-Kolmogorov equations, communicating classes, and stopping times in the Markov context.

2. **Lecture XI: State Classification and Limit Theorems**

   Topics: Transience vs. recurrence, expected return times, invariant distributions, detailed balance (reversibility), and convergence to equilibrium.

3. **Lecture XII: Random Walks and Harmonic Functions**

   Topics: Pólya's recurrence theorem for random walks on $\mathbb{Z}^d$, connection to electrical networks, discrete harmonic functions, and the maximum principle.

1.4 Suggested checkpoints

- Be able to look at a transition graph and immediately identify the transient and recurrent classes, as well as their periods.
- Be able to set up a system of linear equations to solve for the exact probability that a random walk hits state A before state B.